Proceedings of the Fifth Conference of Mathematical Society of Moldova
IMCS-55, September 28 - October 1, 2019, Chisinau, Republic of Moldova

Center-affine invariant stability conditions of unperturbed motion governed by critical cubic differential system

Natalia Neagu, Victor Orlov, Mihail Popa

Abstract

Center-affine invariant Lyapunov stability conditions of unperturbed motion governed by critical cubic differential system were obtained.

Keywords: differential system, comitant, transvectant, Lyapunov stability.

1 Introduction

The cubic differential system $s(1,2,3)$ is of particular interest not only in theoretical, but in a number of applied problems of differential equations. Therefore, the obtaining Lyapunov stability [1] invariant conditions for unperturbed motion governed by the differential system $s(1,2,3)$ is relevant. But, this problem turned out to be very difficult. We couldn't find the regularity in construction of Lyapunov series coefficients, which play an important role in determination of stability of unperturbed motion governed by differential systems $s(1,2,3)$ in critical case. Then we decided to examine the different particular cases of the differential system $s(1,2,3)$: when quadratic part or cubic part have the Darboux form [2]. This allowed us to understand how the problem of stability of unperturbed motion governed by the critical differential systems $s(1,2,3)$ can be solved in general terms.

2 Invariants and comitants of cubic differential system

Let us consider the system of differential equations

$$\frac{dx}{dt}=\sum_{i=1}^{3}P_i(x,y),\qquad \frac{dy}{dt}=\sum_{i=1}^{3}Q_i(x,y), \tag{1}$$

where $P_i(x,y)$, $Q_i(x,y)$ are homogeneous polynomials of degree $i=1,2,3$ in the phase variables x and y. Coefficients and variables in (1) are given over the field of real numbers $\mathbb{R}$.

Let f and φ be two center-affine comitants [3] of the system (1) of degree r and ρ respectively in the phase variables x and y. According to [4], [5] and [6] the polynomial

$$(f,\varphi)^{(k)}=\frac{(r-k)!(\rho-k)!}{r!\rho!}\sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\frac{\partial^k\varphi}{\partial x^h\partial y^{k-h}} \tag{2}$$

also is a center-affine comitant of the system (1) and is called a transvectant of order k in polynomials f and φ.

The following $GL(2;\mathbb{R})$-comitants [3] have the first degree with respect to the coefficients of the system (1):

$$\begin{gathered}R_i=P_i(x,y)y-Q_i(x,y)x,\quad (i=1,2,3),\\ S_i=\frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x}+\frac{\partial Q_i(x,y)}{\partial y}\right),\quad (i=1,2,3),\end{gathered} \tag{3}$$

Using the comitants (3) and the transvectant (2), in [7, 8] the following invariants and comitants of the system (1) were constructed:

$$\begin{gathered}K_1=S_1,\ K_2=R_1,\ K_3=(R_1,R_1)^{(2)},\ K_4=R_2,\ K_5=S_2,\\ K_6=(R_2,R_1)^{(1)},\ K_7=(R_2,R_1)^{(2)},\ K_8=R_3,\ K_9=(R_3,R_1)^{(1)},\\ K_{10}=(R_3,R_1)^{(2)},\ K_{11}=(K_{10},R_1)^{(1)},\ K_{12}=(K_{10},R_1)^{(2)},\\ K_{13}=(K_7,R_1)^{(1)},\ K_{14}=(S_2,R_1)^{(1)},\ K_{15}=S_3,\\ K_{16}=(S_3,R_1)^{(1)},\ K_{17}=(S_3,R_1)^{(2)}.\end{gathered} \tag{4}$$

Later on, we will need the following comitants and invariants of system (1):

$$M_1 = 3K_1K_7 - 4K_1K_{14} + 2K_1^2K_5 - 6K_{13},$$

$$M_2 = -8K_1K_{11} + 6K_1K_2K_{17} + 4K_1^2K_{10} - 6K_1^2K_{16} + 3K_1^3K_{15} + 8K_2K_{12},$$

$$M_3 = 6K_1K_2K_7 + 4K_1K_2K_{14} - 2K_1^2K_2K_5 + 6K_1^2K_6 + 3K_1^3K_4 + 12K_2K_{13},$$

$$M_4 = -K_1K_5 + 2K_{14}, \quad M_5 = 3K_1K_5 + 6K_7 - 2K_{14},$$

$$M_6 = 4K_1K_2K_{11} - 4K_1^2K_2K_{10} + 2K_1^3K_9 - K_1^4K_8 - 2K_2^2K_{12},$$

$$M_7 = K_1K_{17} + 2K_{12}, \quad M_8 = -3K_1K_2K_7 + 2K_1^2K_6 - K_1^3K_4 + 2K_2K_{13},$$

$$M_9 = -3K_1K_7 + 4K_1K_{14},$$

$$M_{10} = 2K_1K_2K_{17} + 8K_1K_{11} + 4K_1^2K_{10} + 2K_1^2K_{16} + K_1^3K_{15} + 8K_2K_{12},$$

$$M_{11} = 2K_1K_2K_{17} + 8K_1K_{11} - 4K_1^2K_{10} - 2K_1^2K_{16} + K_1^3K_{15} - 8K_2K_{12},$$

$$M_{12} = 4K_1K_2K_{11} + 4K_1^2K_2K_{10} + 2K_1^3K_9 + K_1^4K_8 + 2K_2^2K_{12},$$

$$M_{13} = K_1K_{17} - 2K_{12},$$

$$M_{14} = -8K_1K_{11} + 6K_1K_2K_{17} - 4K_1^2K_{10} + 6K_1^2K_{16} + 3K_1^3K_{15} - 8K_2K_{12},$$

$$M_{15} = -8K_1K_{11} + 6K_1K_2K_{17} + 4K_1^2K_{10} + 3K_1^3K_{15} - 6K_1^2K_{16} + 8K_2K_{12}. \tag{5}$$

were K_i $(i = \overline{1, 17})$ are from (4).

3 Invariant stability conditions of unperturbed motion

In [9, 10] the stability conditions of unperturbed motion for critical system (1) expressed through coefficients of this system were provided. Using these results, all cases of critical system (1) are separated by two sets of center-affine conditions:

$$\begin{aligned} &\text{a) } S_1 < 0,\ S_1^2 + 2K_3 = 0,\ M_8 \not\equiv 0; \\ &\text{b) } S_1 < 0,\ S_1^2 + 2K_3 = 0,\ M_8 \equiv 0. \end{aligned} \tag{6}$$

Taking into account the center-affine invariants and comitants (3)-(5), all invariant stability conditions of unperturbed motion for system (1) in the case a) from (6) are obtained. These conditions are large therefore not included here. We will bring here results obtained in case b) from (6).

Theorem 1. *If $S_1 < 0$, $S_1^2 + 2K_3 = 0$ and $M_8 \equiv 0$, then the stability of the unperturbed motion described by system* (1) *is characterized by one of the following 13 possible cases:*
I. $M_1 \not\equiv 0$, *then the unperturbed motion is unstable;*
II. $M_1 \equiv 0$, $M_{15} < 0$, *then the unperturbed motion is unstable;*
III. $M_1 \equiv 0$, $M_{15} > 0$, *then the unperturbed motion is stable;*
IV. $M_1 \equiv M_{15} \equiv 0$, $M_5 M_6 \not\equiv 0$, *then the unperturbed motion is unstable;*
V. $M_1 \equiv M_{15} \equiv M_5 \equiv 0$, $M_6 M_7 > 0$, *then the unperturbed motion is unstable;*
VI. $M_1 \equiv M_{15} \equiv M_5 \equiv 0$, $M_6 M_7 < 0$, *then the unperturbed motion is stable;*
VII. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv 0$, $M_3 M_6 \not\equiv 0$, *then the unperturbed motion is unstable;*
VIII. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv 0$, $M_6 \not\equiv 0$, $M_{10} < 0$, *then the unperturbed motion is unstable;*
IX. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv 0$, $M_6 \not\equiv 0$, $M_{10} > 0$, *then the unperturbed motion is stable;*
X. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv M_{10} \equiv 0$, $M_6 M_{12} > 0$, *then the unperturbed motion is unstable;*
XI. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv M_{10} \equiv 0$, $M_6 M_{12} < 0$, *then the unperturbed motion is stable;*
XII. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv M_{10} \equiv M_{12} \equiv 0$, $M_6 \not\equiv 0$, *then the unperturbed motion is stable;*
XIII. $M_1 \equiv M_{15} \equiv M_6 \equiv 0$, *then the unperturbed motion is stable.*

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motion. Moreover, for sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series.

The expressions S_1, K_3, M_1, M_3, M_5, M_6, M_7, M_{10}, M_{12}, M_{15}, *are from* (3)-(5).

Acknowledgments. This research was partially supported by grants 15.817.02.03F, 15.817.02.18F and 16.80012.02.01F.

References

[1] A.M. Lyapunov. *Obshchaia zadacha ob ustoichivosti dvijenia.* Sobranie sochinenii, II– Moskva-Leningrad: Izd. Acad. Nauk SSSR, 1956 (in Russian).

[2] N. Neagu, V. Orlov, M. Popa. *Invariant conditions of stability of unperturbed motion described by cubic differential system with quadratic part of Darboux type.* The 26th Conference on Applied and Industrial Mathematics (CAIM 2018), 20-23 September, 2018, Technical University of Moldova, Chişinău, Moldova, pp.37–39.

[3] K. S. Sibirsky. *Introduction to the Algebraic Theory of Invariants of Differential Equations.* Manchester University Press, 1988.

[4] N. I. Vulpe. *Polynomial bases of comitants of differential systems and their applications in qualitative theory.* Kishinev, Ştiinţa, 1986 (in Russian).

[5] G. B. Gurevich. *Foundations of the Theory of Algebraic Invariants.* Noordhoff, Groningen, 1964.

[6] D. Boularas, Iu. Calin, L. Timochouk, N. Vulpe. *T-comitants of quadratic systems: A study via the translation invariants*, Report 96-90, Delft University of Technology, Faculty of Technical Mathematics and Informatics, 1996, pp. 1–36.

[7] Iu. Calin. *On rational bases of* $GL(2, R)$*-comitants of planar polynomial systems of differential equations.* Bul. Acad. Ştiinţe Repub. Mold. Mat., 2003, no. 2(42), pp. 69–86.

[8] Iu. Calin, V. Orlov. *A rational basis of $GL(2, R)$-comitants for the bidimensional polynomial system of differential equations of the fifth degree.* The 26th Conference on Applied and Industrial Mathematics (CAIM 2018), 20-23 September, 2018, Technical University of Moldova, Chişinău, Moldova, pp. 27–29.

[9] N. Neagu, V. Orlov, M. Popa. *Invariant stability conditions of unperturbed motion governed by critical differential systems $s(1, 2, 3)$ in a nonsingular case.* International Conference "Mathematics & IT: Research and Education" (MITRE-2019), Chisinau, Republic of Moldova, June 24-26, 2019, pp. 49–51.

[10] N. Neagu, V. Orlov, M. Popa. *Invariant stability conditions of unperturbed motion governed by critical differential system $s(1, 2, 3)$ on a singular invariant manifold.* The 27th Conference on Applied and Industrial Mathematics (CAIM 2019), 19-23 September, 2019, "Valahia" University of Târgovişte, Târgovişte, Romania, 2p, (accepted).

Natalia Neagu[1], Victor Orlov[2], Mihail Popa[3]

[1]Tiraspol State University and
Ion Creangă State Pedagogical University
E–mail: neagu_natusik@mail.ru

[2]Technical University of Moldova and
Vladimir Andrunachievici Institute of Mathematics and Computer Science,
E–mail: orlovictor@gmail.com

[3] Vladimir Andrunachievici Institute of Mathematics and Computer Science and
Tiraspol State University
E–mail: mihailpomd@gmail.com