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NEW 3D THERMOELASTIC INFLUENCE FUNCTIONS, CAUSED BY A UNITARY POINT HEAT SOURCE, APPLIED IN A QUARTER OF LAYER

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Abstract. The aim of this paper consist in the constructing of the main thermoelastic displacements Green's functions (MTDGFs) for a generalized 3D BVP of uncoupled thermoelasticity for a quarter of layer. To reach this aim are derived structural formulas for MTDGFs, expressed via respective Green's functions for Poisson's equation (GFPE) by using harmonic integral representations method (HIRM). These structural formulas are validated by the checking the equations of thermoelasticity with respect to point of response in which the thermoelastic displacements appeared and with respect to point of application the heat source and the nonhomogeneous Poisson's equation. In addition, they satisfy the homogeneous mechanical boundary conditions for MTDGFs with respect to point application the displacements and to mechanical boundary conditions and temperature Green's function with respect to point of application the heat source. The thermoelastic volume dilatation (TVD) derived separately from respective integral representations has been equal to the TVD derived by using structural formulas for MTDGFs. The final analytical expressions for MTDGFs obtained on the base of mentioned above structural formulas for sixteen new 3D BVPs of thermoelasticity within quarter of layer contain Bessel functions of the zero-order of the second type. These results are presented graphically.

Abbreviations

HIRM - harmonic integral representation method;
MTDGFs - main thermoelastic displacements Green's functions;
3D - three dimensional;
BVP – boundary value problem;
GFPE - Green function for Poisson Equation;
HIRs – harmonic integral representations;
GFs - Green's functions;
TVD – thermoelastic volume dilatation;
IFs – integration formulas.

1. Introduction

The obtained in this paper results are considered for uncoupled thermoelasticity, in special for theory of thermal stresses, theories of which are presented in the classical [1]-[7] and modern [8] scientific literature. The TVD, MTDGFs and IFs were derived by using HIRM in the works [9]-[16]. In this paper is proposed the development of the HIRM to derivation of thermoelastic structural formulas for a generalized BVP, which

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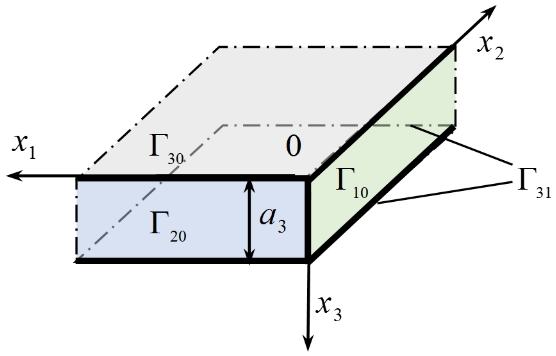

FIGURE 1. The scheme of the quarter of layer $V \equiv (0 \le x_1 < \infty, 0 \le x_2 < \infty, 0 \le x_3 \le a_3)$ with boundary quadrants $\Gamma_{30}((0 \le x_1 < \infty, 0 \le x_2 < \infty, x_3 = 0))$, $\Gamma_{31}(0 \le x_1 < \infty, 0 \le x_2 < \infty, x_3 = a_3)$ and with boundary half-strips $\Gamma_{10}(x_1 = 0, 0 \le x_2 < \infty, 0 \le x_3 \le a_3)$ and $\Gamma_{20}(0 \le x_1 < \infty, x_2 = 0, 0 \le x_3 \le a_3)$.

will permit to the readers to obtain analytical expressions for TVD, MTDGFs and IFs for sixteen BVPs for the quarter of layer (Figure 1).

Objectives

The main objective of this paper is to develop HIRM in such a way that the readers will be able to derive the analytical expressions for TVD, MTDGFs and IFs for sixteen new locally-mixed 3D BVPs of thermoelasticity within the quarter of layer V.

2. Formulation of the generalized BVP for thermoelastic quarter of layer

The generalized BVP to uncoupled thermoelasticity for determining structural formulas for MTDGFs for displacements $U_i(x,\xi); i = 1,2,3$ within the quarter of layer consist from Lame's and Poisson's equations:

$$\begin{aligned} &\mu \bigtriangledown_x^2 U_i(x,\xi) + (\lambda+\mu)\Theta_{,x_i}(x,\xi) - \gamma G_{T,x_i}(x,\xi) = 0; i = 1,2,3; \\ &\nabla_x^2 G_T(x,\xi) = -\delta(x-\xi); x \equiv (x_1,x_2,x_3), \xi \equiv (\xi_1,\xi_2,\xi_3), \end{aligned} \tag{1}$$

where

$\bigtriangledown_x^2$ - 3D Laplace differential operator; λ, μ - Lame's constants of elasticity; $\Theta = U_{x_j,x_j}(x,\xi)$ - TVD; $\gamma = \alpha(3\lambda + 2\mu)$ is the thermoelastic constant, α is the coefficient of linear temperature dilatation.

In addition, MTDGFs have to satisfy the following sixteen possible combinations of the boundary conditions for $U_i(y,\xi), i = 1,2,3$, thermal stresses $\sigma^*_{ij}(y,\xi); i,j = 1,2,3$, and GF $G_T(y,\xi)$ for temperature or its derivative on external normal $\partial G_T(y,\xi)/\partial n_\Gamma$:

$$\begin{aligned} U_1(y,\xi) = \sigma^*_{12}(y,\xi) = \sigma^*_{13}(y,\xi) = 0, \partial G_T(y,\xi)/\partial n_{\Gamma_{10}} = 0; \\ \xi \in V; y \equiv (0,y_2,y_3) \in \Gamma_{10}; \end{aligned} \tag{2}$$

or

$$\sigma^*_{11}(y,\xi) = U_2(y,\xi) = U_3(y,\xi) = 0, G_T(y,\xi) = 0; \xi \in V; y \equiv (0,y_2,y_3) \in \Gamma_{10}; \tag{3}$$

on the boundary half-strip $\Gamma_{10}(y_1 = 0, 0 \le y_2 < \infty, 0 \le y_3 \le a_3)$, and

$$U_2(y,\xi)=\sigma^*_{21}(y,\xi)=\sigma^*_{23}(y,\xi)=0, \partial G_T(y,\xi)/\partial n_{\Gamma_{20}}=0;\\ \xi\in V; y\equiv(y_1,0,y_3)\in\Gamma_{20}; \tag{4}$$

or

$$\sigma^*_{22}(y,\xi)=U_1(y,\xi)=U_3(y,\xi)=0, G_T(y,\xi)=0; \xi\in V; y\equiv(y_1,0,y_3)\in\Gamma_{20}; \tag{5}$$

on the boundary half-strip $\Gamma_{20}(0\le y_1<\infty, y_2=0, 0\le y_3\le a_3)$.

$$\sigma^*_{31}(y,\xi)=\sigma^*_{32}(y,\xi)=U_3(y,\xi)=0, \partial G_T(y,\xi)/\partial n_{\Gamma_{30}}=0;\\ \xi\in V; y\equiv(y_1,y_2,0)\in\Gamma_{30}; \tag{6}$$

or

$$U_1(y,\xi)=U_2(y,\xi)=\sigma^*_{33}(y,\xi)=0, G_T(y,\xi)=0; \xi\in V; y\equiv(y_1,y_2,0)\in\Gamma_{30}; \tag{7}$$

on the boundary quadrant $\Gamma_{30}(0\le y_1<\infty, 0\le y_2<\infty, y_3=0)$.

$$\sigma^*_{31}(y,\xi)=\sigma^*_{32}(y,\xi)=U_3(y,\xi)=0, \partial G_T(y,\xi)/\partial n_{\Gamma_{31}}=0;\\ \xi\in V; y\equiv(y_1,y_2,a_3)\in\Gamma_{31}; \tag{8}$$

or

$$U_1(y,\xi)=U_2(y,\xi)=\sigma^*_{33}(y,\xi)=0, G_T(y,\xi)=0; \xi\in V; y\equiv(y_1,y_2,a_3)\in\Gamma_{31}; \tag{9}$$

on the boundary quadrant $\Gamma_{31}(0\le y_1<\infty, 0\le y_2<\infty, y_3=a_3)$.

3. General integral representations for TVD and MTDGFs

To derive the structural formulas for TVD and MTDGFs we use the following general integral representations of HIRM [10]-[13].

$$\Theta(x,\xi)=\frac{\gamma}{\lambda+2\mu}G_\Theta(x,\xi)+\int\limits_\Gamma\left[\frac{\partial\Theta(y,\xi)}{\partial n_\Gamma}-\Theta(y,\xi)\frac{\partial}{\partial n_\Gamma}\right]G_\Theta(x,y)d\Gamma(y); \tag{10}$$

- for TVD Θ, and

$$U_i(x,\xi)=\frac{\gamma x_i}{2\mu}G_T(x,\xi)-\frac{\lambda+\mu}{2\mu}x_i\Theta(x,\xi)-\frac{\gamma}{2(\lambda+2\mu)}\xi_iG_i(x,\xi) \tag{11}$$

$$+\int\limits_\Gamma\left[G_i(x,y)\frac{\partial}{\partial n_\Gamma}-\frac{\partial G_i(x,y)}{\partial n_\Gamma}\right]\left[U_i(y,\xi)+\frac{y_i}{2\mu}\left[(\lambda+\mu)\Theta(y,\xi)-\gamma G_T(y,\xi)\right]\right]d\Gamma(y);$$

$$i=1,2,3;$$

- for MTDGFs U_i.

The general integral representations (10) and (11) in the particular case of quarter of layer V can be rewritten as following:

$$\Theta(x,\xi)=\frac{\gamma}{\lambda+2\mu}G_\Theta(x,\xi)+\sum_{k=1}^{3}\int\limits_{\Gamma_{k0}}\left[\frac{\partial\Theta(y,\xi)}{\partial n_{\Gamma_{k0}}}-\Theta(y,\xi)\frac{\partial}{\partial n_{\Gamma_{k0}}}\right]G_\Theta(x,y)d\Gamma_{k0}(y)$$

$$+\int\limits_{\Gamma_{31}}\left[\frac{\partial\Theta(y,\xi)}{\partial n_{\Gamma_{31}}}-\Theta(y,\xi)\frac{\partial}{\partial n_{\Gamma_{31}}}\right]G_\Theta(x,y)d\Gamma_{31}(y); \tag{12}$$

- for TVD Θ, and

$$U_i(x,\xi)=\frac{\gamma x_i}{2\mu}G_T(x,\xi)-\frac{\lambda+\mu}{2\mu}x_i\Theta(x,\xi)-\frac{\gamma}{2(\lambda+2\mu)}\xi_i G_i(x,\xi)$$

$$-\sum_{k=1}^{3}\int\limits_{\Gamma_{k0}}\left[\frac{\partial G_i(x,y)}{\partial n_{\Gamma_{k0}}}-G_i(x,y)\frac{\partial}{\partial n_{\Gamma_{k0}}}\right] \tag{13}$$

$$\times\left[U_i(y,\xi)+\frac{y_i}{2\mu}\left[(\lambda+\mu)\Theta(y,\xi)-\gamma G_T(y,\xi)\right]\right]d\Gamma_{k0}(y)$$

$$-\int\limits_{\Gamma_{31}}\left[\frac{\partial G_i(x,y)}{\partial n_{\Gamma_{31}}}-G_i(x,y)\frac{\partial}{\partial n_{\Gamma_{31}}}\right]\left[U_i(y,\xi)+\frac{y_i}{2\mu}\left[(\lambda+\mu)\Theta(y,\xi)-\gamma G_T(y,\xi)\right]\right]d\Gamma_{31}(y);$$

$$i=1,2,3;$$

- for MTDGFs U_i.

In addition, the boundary conditions (2) - (9) can be transformed in equivalent boundary conditions if it is taking into account the following links between displacements U_i, stresses σ^*_{ij}, TVD Θ and respective GFPE G_i, G_Θ on boundary half-strips Γ_{10},Γ_{20} and on boundary quadrants $\Gamma_{3i}; i=0;1$ of the quarter of layer V. So, if $U_i=0$, then $G_i=0$; if $U_{i,n}=0$, then $G_{i,n}=0$. Also, if zero normal displacements, zero tangential stresses and $G_{T,n}=0$ are given, then $\Theta_{,n}=0$ and $G_{\Theta,n}=0$. Finally, if zero normal stresses, zero tangential displacements and $G_T=0$, then $\Theta=0$ and $G_\Theta=0$ [9]-[12], [14], [15]. Also, if temperature T or heat flux $\alpha\partial T/\partial n$ are given on the marginal planes or their parts (straight lines or their parts), then on these planes or their parts (straight lines or their parts) $G_T=0$, or $\partial G_T/\partial n=0$, respectively. In such a way, the transformed boundary conditions (2) - (9) looks as following equivalent conditions:

$$\begin{gathered}U_1(y,\xi)=\sigma^*_{12}(y,\xi)=\sigma^*_{13}(y,\xi)=0, G_{T,y_1}(y,\xi)=0\Rightarrow\\ U_1(y,\xi)=U_{1,y_3}(y,\xi)=U_{3,y_1}(y,\xi)=U_{1,y_2}(y,\xi)=U_{2,y_1}(y,\xi)=0\Rightarrow\\ \Theta_{,y_1}(y,\xi)=G_{1,y_3}(y,\xi)=G_{2,y_1}(y,\xi)=G_1(y,\xi)=G_{\Theta,y_1}(y,\xi)=\\ G_{T,y_1}(y,\xi)=0;\xi\in V; y\equiv(0,y_2,y_3)\in\Gamma_{10};\end{gathered}\tag{14}$$

or

$$\begin{gathered}\sigma^*_{11}(y,\xi)=U_2(y,\xi)=U_3(y,\xi)=0, G_T(y,\xi)=0\Rightarrow\\ U_3(y,\xi)=U_{3,y_3}(y,\xi)=U_{3,y_2}(y,\xi)=U_{2,y_2}(y,\xi)=U_{2,y_3}(y,\xi)=0\Rightarrow\\ \Theta(y,\xi)=G_{1,y_1}(y,\xi)=G_2(y,\xi)=G_3(y,\xi)=G_\Theta(y,\xi)=\\ G_T(y,\xi)=0;\xi\in V; y\equiv(0,y_2,y_3)\in\Gamma_{10};\end{gathered}\tag{15}$$

- on the boundary half-strip $\Gamma_{10}(y_1=0, 0\le y_2<\infty, 0\le y_3\le a_3)$, and

$$\begin{gathered}U_2(y,\xi)=\sigma^*_{21}(y,\xi)=\sigma^*_{23}(y,\xi)=0, G_{T,y_2}(y,\xi)=0\Rightarrow\\ U_2(y,\xi)=U_{2,y_3}(y,\xi)=U_{3,y_2}(y,\xi)=U_{1,y_2}(y,\xi)=U_{2,y_1}(y,\xi)=0\Rightarrow\\ \Theta_{,y_2}(y,\xi)=G_{2,y_3}(y,\xi)=G_{3,y_2}(y,\xi)=G_2(y,\xi)=G_{\Theta,y_2}(y,\xi)=\\ G_{T,y_2}(y,\xi)=0;\xi\in V; y\equiv(y_1,0,y_3)\in\Gamma_{20};\end{gathered}\tag{16}$$

or

$$\begin{gathered}\sigma^*_{22}(y,\xi)=U_1(y,\xi)=U_3(y,\xi)=0, G_T(y,\xi)=0\Rightarrow\\ U_3(y,\xi)=U_{2,y_2}(y,\xi)=U_{3,y_1}(y,\xi)=U_{1,y_1}(y,\xi)=U_{1,y_3}(y,\xi)=0\Rightarrow\end{gathered}$$

$$\Theta(y,\xi) = G_1(y,\xi) = G_3(y,\xi) = G_{2,y_2}(y,\xi) = G_\Theta(y,\xi) = \\ G_T(y,\xi) = 0; \xi \in V; y \equiv (y_1, 0, y_3) \in \Gamma_{20}; \tag{17}$$

- on the boundary half-strip $\Gamma_{20}(0 \le y_1 < \infty, y_2 = 0, 0 \le y_3 \le a_3)$,

$$U_3(y,\xi) = \sigma^*_{31}(y,\xi) = \sigma^*_{32}(y,\xi) = 0, G_{T,y_3}(y,\xi) = 0 \Rightarrow \\ U_3(y,\xi) = U_{1,y_3}(y,\xi) = U_{3,y_1}(y,\xi) = U_{3,y_2}(y,\xi) = U_{2,y_3}(y,\xi) = 0 \Rightarrow \\ \Theta_{,y_3}(y,\xi) = G_{1,y_3}(y,\xi) = G_{2,y_3}(y,\xi) = G_3(y,\xi) = G_{\Theta,y_3}(y,\xi) = \\ G_{T,y_3}(y,\xi) = 0; \xi \in V; y \equiv (y_1, y_2, 0) \in \Gamma_{30}; \tag{18}$$

or

$$\sigma^*_{33}(y,\xi) = U_1(y,\xi) = U_2(y,\xi) = 0, G_T(y,\xi) = 0 \Rightarrow \\ U_1(y,\xi) = U_{1,y_1}(y,\xi) = U_{1,y_2}(y,\xi) = U_{3,y_3}(y,\xi) = U_{2,y_1}(y,\xi) = U_{2,y_2}(y,\xi) = 0 \Rightarrow \\ \Theta(y,\xi) = G_1(y,\xi) = G_{3,y_3}(y,\xi) = G_2(y,\xi) = G_\Theta(y,\xi) = \\ G_T(y,\xi) = 0; \xi \in V; y \equiv (y_1, y_2, 0) \in \Gamma_{30}; \tag{19}$$

- on the boundary quadrant $\Gamma_{30}(0 \le y_1 < \infty, 0 \le y_2 < \infty, y_3 = 0)$;

$$U_3(y,\xi) = \sigma^*_{31}(y,\xi) = \sigma^*_{32}(y,\xi) = 0, G_{T,y_3}(y,\xi) = 0 \Rightarrow \\ U_3(y,\xi) = U_{1,y_3}(y,\xi) = U_{3,y_1}(y,\xi) = U_{3,y_2}(y,\xi) = U_{2,y_3}(y,\xi) = 0 \Rightarrow \\ \Theta_{,y_3}(y,\xi) = G_{1,y_3}(y,\xi) = G_{2,y_3}(y,\xi) = G_3(y,\xi) = G_{\Theta,y_3}(y,\xi) = \\ G_{T,y_3}(y,\xi) = 0; \xi \in V; y \equiv (y_1, y_2, a_3) \in \Gamma_{31}; \tag{20}$$

or

$$\sigma^*_{33}(y,\xi) = U_1(y,\xi) = U_2(y,\xi) = 0, G_T(y,\xi) = 0 \Rightarrow \\ U_1(y,\xi) = U_{1,y_1}(y,\xi) = U_{1,y_2}(y,\xi) = U_{3,y_3}(y,\xi) = U_{2,y_1}(y,\xi) = U_{2,y_2}(y,\xi) = 0 \Rightarrow \\ \Theta(y,\xi) = G_1(y,\xi) = G_{3,y_3}(y,\xi) = G_2(y,\xi) = G_\Theta(y,\xi) = \\ G_T(y,\xi) = 0; \xi \in V; y \equiv (y_1, y_2, a_3) \in \Gamma_{31}; \tag{21}$$

- on the boundary quadrant $\Gamma_{31}(0 \le y_1 < \infty, 0 \le y_2 < \infty, y_3 = a_3)$, where

$$\sigma^*_{ij} = \mu(U_{i,j} + U_{j,i}) + \delta_{ij}(\lambda\Theta - \gamma G_T), \tag{22}$$

and δ_{ij} is the Kronecker's symbol and G_T is GFPE for temperature.

Taken into account the equivalent boundary conditions (14) - (21) for Θ and G_Θ , from integral representation (12), follows that the TVD $\Theta(x,\xi)$ is written in the form:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda + 2\mu} G_\Theta(x,\xi). \tag{23}$$

This can be explained by the fact that for each boundary conditions on half-strips Γ_{10}, Γ_{20} (14) - (17) and on quadrants Γ_{30}, Γ_{31} (18) - (21), for any combinations of the boundary conditions (14) - (21), the integrals in Eq. (12) vanish. Therefore, the final formula for TVD looks as in Eq. (23). But, from boundary conditions (14) - (21) for G_Θ and G_T follows $G_\Theta(x,\xi) = G_T(x,\xi)$. This last result and Eq. (23) leads to the following final structural formula for TVD:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda + 2\mu} G_T(x,\xi). \tag{24}$$

4. Structural formulas for MTDGFs

Substituting, the structural formula (24) and boundary conditions (14) - (21) into Eq. (13), it is seen that these integral representations can be simplified substantially, therefore we obtain the following structural formulas for MTDGFs:

$$U_1(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)}\left[x_1 G_T(x,\xi) - \xi_1 G_1(x,\xi)\right]; \tag{25}$$

$$U_2(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)}\left[x_2 G_T(x,\xi) - \xi_2 G_2(x,\xi)\right]; \tag{26}$$

$$\begin{aligned} U_3(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)}\Bigg\{ & x_3 G_T(x,\xi) - \xi_3 G_3(x,\xi) \\ & + a_3 \int\limits_{\Gamma_{31}} \left(\frac{\partial G_3(x,y)}{\partial n_{\Gamma_{31}}} - G_3(x,y)\frac{\partial}{\partial n_{\Gamma_{31}}}\right) G_T(y,\xi)\, d\Gamma_{31}(y)\Bigg\}. \end{aligned} \tag{27}$$

The integral in Eq. (27) can be calculated as follows:

$$\begin{aligned} I_3(x,\xi) &= a_3 \int\limits_{\Gamma_{31}} \left(\frac{\partial G_3(x,y)}{\partial n_{\Gamma_{31}}} - G_3(x,y)\frac{\partial}{\partial n_{\Gamma_{31}}}\right) G_T(y,\xi)\, d\Gamma_{31}(y) \\ &= \xi_3 G_3(x,\xi) - x_3 G_T(x,\xi) \end{aligned} \tag{28}$$

$$-\int \left[x_1 G_{T,x_1}(x,\xi) - \xi_1 G_{1,x_1}(x,\xi)\right] dx_3 - \int \left[x_2 G_{T,x_2}(x,\xi) - \xi_2 G_{2,x_2}(x,\xi)\right] dx_3.$$

Thus, substituting (28) in (27), we obtain the final structural formula for $U_3(x,\xi)$ in the form:

$$\begin{aligned} U_3(x,\xi) = -\frac{\gamma}{2(\lambda+2\mu)} \int & \left[x_1 G_{T,x_1}(x,\xi)\right. \\ & \left. -\xi_1 G_{1,x_1}(x,\xi) + x_2 G_{T,x_2}(x,\xi) - \xi_2 G_{2,x_2}(x,\xi)\right] dx_3, \end{aligned} \tag{29}$$

in such a way that the TVD calculated on the basis of structural formulas (24), (25) and (29):

$$\Theta(x,\xi) = U_{i,i}(x,\xi) = \frac{\gamma}{\lambda+2\mu} G_T(x,\xi), \tag{30}$$

coincides to the respective TVD given by equation (24), which was calculated independently by using integral representations (12), boundary conditions (14)-(21) for $\Theta(x,\xi)$ and GFPE $G_\Theta(x,\xi)$, $G_T(x,\xi)$.

Also, the validation of the obtained structural formulas for MTDGFs (25), (26), (29) and TVD (24) by using HIRM [9]-[16] was done by using the main formula of $G\Theta$-convolution method [19]:

$$U_i(x,\xi) = \gamma \int\limits_V G_T(z,\xi)\Theta^{(i)}(x,z)\, dV(z), \tag{31}$$

where

$\Theta^{(i)}(x,z)$ is the influence functions for volume dilatation, created by a unit point force, presented for different BVPs in handbook [17].

Thus, according to the formula (31), the validation of the obtained structural formulas for MTDGFs (25), (26), (29) and TVD (24) consist in the satisfaction of the following additional BVPs:

1) With respect to the point $x \equiv (x_1, x_2, x_3)$ MTDGFs $U_i(x, \xi)$ have to satisfy generalized BVP Lame's type equation:

$$\mu \nabla_x^2 U_i(x, \xi) + (\lambda + \mu)\Theta_{,x_i}(x, \xi) - \gamma G_{T,x_i}(x, \xi) = 0, \tag{32}$$

and mechanical boundary conditions (2) - (9).

2) With respect to the point $\xi \equiv (\xi_1, \xi_2, \xi_3)$ MTDGFs $U_i(x, \xi)$ have to satisfy generalized BVP, which consist from equation:

$$\nabla_\xi^2 U_i(x, \xi) = -\gamma \Theta^{(i)}(x, \xi), \tag{33}$$

and thermal boundary conditions (2) - (9).

To note that the obtained structural formulas for MTDGFs (25), (26), (29) and TVD (24) were proved for generalized BVPs, which consist from Eq. (1) and every combination (from sixteen possible) boundary conditions (2) - (9) or (14) - (21).

5. An example of derivation analytical expressions for TVD $\Theta(x, \xi)$ and MTDGFs $U_i(x, \xi)$

To derive analytical expressions for TVD $\Theta(x, \xi)$ and MTDGFs $U_i(x, \xi)$ we have to use the respective structural formulas for TVD (24), for MTDGFs (25), (26), (29) and analytical expressions for GFPE: G_T and G_1, G_2, G_3 in according to respective combination of boundary conditions (14) - (21) and the handbook [17] (see the Problems 20.P.1 - 20.16 and answers to them).

Example. To derive analytical expressions for TVD and MTDGFs inside of the thermoelastic quarter of layer for the BVP, which consist from Eq. (1) and boundary conditions (2), (4), (6), and (8) or (14), (16), (18) and (20).

Solution:

a) In according to respective combination of boundary conditions (2), (4), (6), and (8) or (14), (16), (18) and (20) and the handbook [17] (see Problems: P.20.6, P.20.14, P.20.10, P.20.2) we obtain the following analytical GFPE G_1, G_2, G_3 and G_T:

$$G_1 = G^{(6)}; G_2 = G^{(14)}; G_3 = G^{(10)}; G_T = G^{(2)}; \tag{34}$$

b) We rewrite structural formulas for TVD (24), for MTDGFs (25), (26) and (29) in the form:

$$\Theta(x, \xi) = \frac{\gamma}{\lambda + 2\mu} G_T(x, \xi) = \frac{\gamma}{\lambda + 2\mu} G^{(2)}(x, \xi); \tag{35}$$

$$\begin{aligned} U_1(x, \xi) &= \frac{\gamma}{2(\lambda + 2\mu)} \left[x_1 G_T(x, \xi) - \xi_1 G_1(x, \xi) \right] \\ &= \frac{\gamma}{2(\lambda + 2\mu)} \left[x_1 G^{(2)}(x, \xi) - \xi_1 G^{(6)}(x, \xi) \right]; \end{aligned} \tag{36}$$

$$\begin{aligned} U_2(x, \xi) &= \frac{\gamma}{2(\lambda + 2\mu)} \left[x_2 G_T(x, \xi) - \xi_2 G_2(x, \xi) \right] \\ &= \frac{\gamma}{2(\lambda + 2\mu)} \left[x_2 G^{(2)}(x, \xi) - \xi_2 G^{(14)}(x, \xi) \right]; \end{aligned} \tag{37}$$

$$U_3(x,\xi) = -\frac{\gamma}{2(\lambda+2\mu)} \int [x_1 G_{T,x_1}(x,\xi)$$
$$-\xi_1 G_{1,x_1}(x,\xi) + x_2 G_{T,x_2}(x,\xi) - \xi_2 G_{2,x_2}(x,\xi)]\, dx_3$$
$$= -\frac{\gamma}{2(\lambda+2\mu)} \int \Big[x_1 G^{(2)}_{,x_1}(x,\xi)$$
$$-\xi_1 G^{(6)}_{,x_1}(x,\xi) + x_2 G^{(2)}_{,x_2}(x,\xi) - \xi_2 G^{(14)}_{,x_2}(x,\xi)\Big]\, dx_3; \tag{38}$$

c) From handbook [17] (see Answers to Problems: P.20.6, P.20.14, P.20.10, P.20.2) we rewrite the following analytical expressions for GFPE $G^{(2)}, G^{(6)}, G^{(10)}, G^{(14)}$:

$$G^{(2)}(x,\xi) = \frac{1}{\pi a_3} \sum_{n=1}^{\infty} [K_0(\mu_1 r) \tag{39}$$
$$+K_0(\mu_1 r_1) + K_0(\mu_1 r_2) + K_0(\mu_1 r_{12})] \cos\mu_1 x_3 \cos\mu_1 \xi_3;\, \mu_1 = \frac{n\pi}{a_3};$$
$$G^{(6)}(x,\xi) = \frac{1}{\pi a_3} \sum_{n=1}^{\infty} [K_0(\mu_1 r) \tag{40}$$
$$-K_0(\mu_1 r_1) + K_0(\mu_1 r_2) - K_0(\mu_1 r_{12})] \cos\mu_1 x_3 \cos\mu_1 \xi_3;\, \mu_1 = \frac{n\pi}{a_3};$$
$$G^{(10)}(x,\xi) = \frac{1}{\pi a_3} \sum_{n=1}^{\infty} [K_0(\mu_1 r) \tag{41}$$
$$+K_0(\mu_1 r_1) + K_0(\mu_1 r_2) + K_0(\mu_1 r_{12})] \sin\mu_1 x_3 \sin\mu_1 \xi_3;\, \mu_1 = \frac{n\pi}{a_3};$$
$$G^{(14)}(x,\xi) = \frac{1}{\pi a_3} \sum_{n=1}^{\infty} [K_0(\mu_1 r) \tag{42}$$
$$+K_0(\mu_1 r_1) - K_0(\mu_1 r_2) - K_0(\mu_1 r_{12})] \cos\mu_1 x_3 \cos\mu_1 \xi_3;\, \mu_1 = \frac{n\pi}{a_3},$$

where the functions $K_0(\mu_1 r)$, $K_0(\mu_1 r_1)$, $K_0(\mu_1 r_2)$ and $K_0(\mu_1 r_{12})$ are modified Bessel functions (or cylindrical functions) of the zero-order of the second kind:

$$r = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2};\, r_1 = \sqrt{(x_1+\xi_1)^2 + (x_2-\xi_2)^2};$$

$$r_2 = \sqrt{(x_1-\xi_1)^2 + (x_2+\xi_2)^2};\, r_{12} = \sqrt{(x_1+\xi_1)^2 + (x_2+\xi_2)^2}.$$

d)) Calculation of the analytical expressions for MTDGFs $U_3(x,\xi)$ by using Eq. (38):

$$U_3(x,\xi) = -\frac{\gamma}{2(\lambda+2\mu)} \int \Big[x_1 G^{(2)}_{,x_1}(x,\xi)$$
$$-\xi_1 G^{(6)}_{,x_1}(x,\xi) + x_2 G^{(2)}_{,x_2}(x,\xi) - \xi_2 G^{(14)}_{,x_2}(x,\xi)\Big]\, dx_3$$
$$= \frac{\gamma}{2\pi a_3(\lambda+2\mu)} \sum_{n=1}^{\infty} [r K_1(\mu_1 r) \tag{43}$$
$$+r_1 K_1(\mu_1 r_1) + r_2 K_1(\mu_1 r_2) + r_{12} K_1(\mu_1 r_{12})] \sin\mu_1 x_3 \cos\mu_1 \xi_3;$$

where

$$K_1(\mu_1 r) = -\frac{\partial}{\partial(\mu_1 r)} K_0(\mu_1 r); K_1(\mu_1 r_1) = -\frac{\partial}{\partial(\mu_1 r_1)} K_0(\mu_1 r_1);$$
$$K_1(\mu_1 r_2) = -\frac{\partial}{\partial(\mu_1 r_2)} K_0(\mu_1 r_2); K_1(\mu_1 r_{12}) = -\frac{\partial}{\partial(\mu_1 r_{12})} K_0(\mu_1 r_{12}), \tag{44}$$

are the modified Bessel functions (or cylindrical functions) of the first-order of the second kind.

e) Final analytical expressions for TVD and MTDGFs within thermoelastic a quarter-layer for BVP (1), (2), (4), (6), (8), obtained by using structural formulas (35) - (38) and GFPE (39)-(42) can be written in the form:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda+2\mu} G^{(2)}(x,\xi) = \frac{\gamma}{\pi a_3(\lambda+2\mu)} \sum_{n=1}^{\infty} [K_0(\mu_1 r) \tag{45}$$
$$+K_0(\mu_1 r_1) + K_0(\mu_1 r_2) + K_0(\mu_1 r_{12})] \cos\mu_1 x_3 \cos\mu_1 \xi_3; \mu_1 = \frac{n\pi}{a_3};$$

- for TVD, and

$$U_1(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)} \left[x_1 G^{(2)}(x,\xi) - \xi_1 G^{(6)}(x,\xi)\right] = \frac{\gamma}{2\pi a_3(\lambda+2\mu)}$$
$$\times \sum_{n=1}^{\infty} \{(x_1-\xi_1)[K_0(\mu_1 r) + K_0(\mu_1 r_2)] + (x_1+\xi_1)[K_0(\mu_1 r_1) + K_0(\mu_1 r_{12})]\} \tag{46}$$
$$\times \cos\mu_1 x_3 \cos\mu_1 \xi_3; \mu_1 = \frac{n\pi}{a_3};$$

$$U_2(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)} \left[x_2 G^{(2)}(x,\xi) - \xi_2 G^{(14)}(x,\xi)\right] = \frac{\gamma}{2\pi a_3(\lambda+2\mu)}$$
$$\times \sum_{n=1}^{\infty} \{(x_2-\xi_2)[K_0(\mu_1 r) + K_0(\mu_1 r_1)] + (x_2+\xi_2)[K_0(\mu_1 r_2) + K_0(\mu_1 r_{12})]\} \tag{47}$$
$$\times \cos\mu_1 x_3 \cos\mu_1 \xi_3; \mu_1 = \frac{n\pi}{a_3};$$

$$U_3(x,\xi) = -\frac{\gamma}{2(\lambda+2\mu)} \int \left[x_1 G^{(2)}_{,x_1}(x,\xi)\right.$$
$$\left. - \xi_1 G^{(6)}_{,x_1}(x,\xi) + x_2 G^{(2)}_{,x_2}(x,\xi) - \xi_2 G^{(14)}_{,x_2}(x,\xi)\right] dx_3 = \frac{\gamma}{2\pi a_3(\lambda+2\mu)} \tag{48}$$
$$\times \sum_{n=1}^{\infty} [r K_1(\mu_1 r) + r_1 K_1(\mu_1 r_1) + r_2 K_1(\mu_1 r_2) + r_{12} K_1(\mu_1 r_{12})]$$
$$\times \sin\mu_1 x_3 \cos\mu_1 \xi_3; \mu_1 = \frac{n\pi}{a_3};$$

- for MTDGFs.

To be noted that, if in the equations (45) – (48) will be omitted the terms, which contain r_2 and r_{12}, then we obtain the respective analytical expressions for TVD and MTDGFs within thermoelastic half-layer [18].

6. Graphical presentation of MTDGFs for thermoelastic quarter of layer

Graphs of the thermoelastic displacements $U_i(x, \xi)$ within the thermoelastic quarter of layer V, caused by a unit heat source applied in the point ($\xi \equiv (\xi_1, \xi_2, \xi_3)$) were plotted by using the soft Maple 18 and the following values of the constants: Poisson ratio $\nu = 0,3$; modulus of elasticity $E = 2,1 \cdot 10^5 MPa$ and coefficient of linear thermal dilatation $\alpha = 1,2 \cdot 10^{-5} (K^{-1})$.

Using the exact expressions (46) - (48), the graphs of the MTDGFs $U_i(x, \xi)$ in dependence of x_1, x_3, within the thermoelastic quarter of layer V for $0 \leq x_1 \leq 10m$; $x_2 = 2,1m$; $0 \leq x_3 \leq 2m$, caused by a unit heat source applied in the point $\xi_1 = 5m, \xi_2 = 2, \xi_3 = 1m$ are presented in the Figure 2 ($U_1(x, \xi)$ - Figure 2a); $U_2(x, \xi)$ - Figure 2b) and $U_3(x, \xi)$ - Figure 2c)).

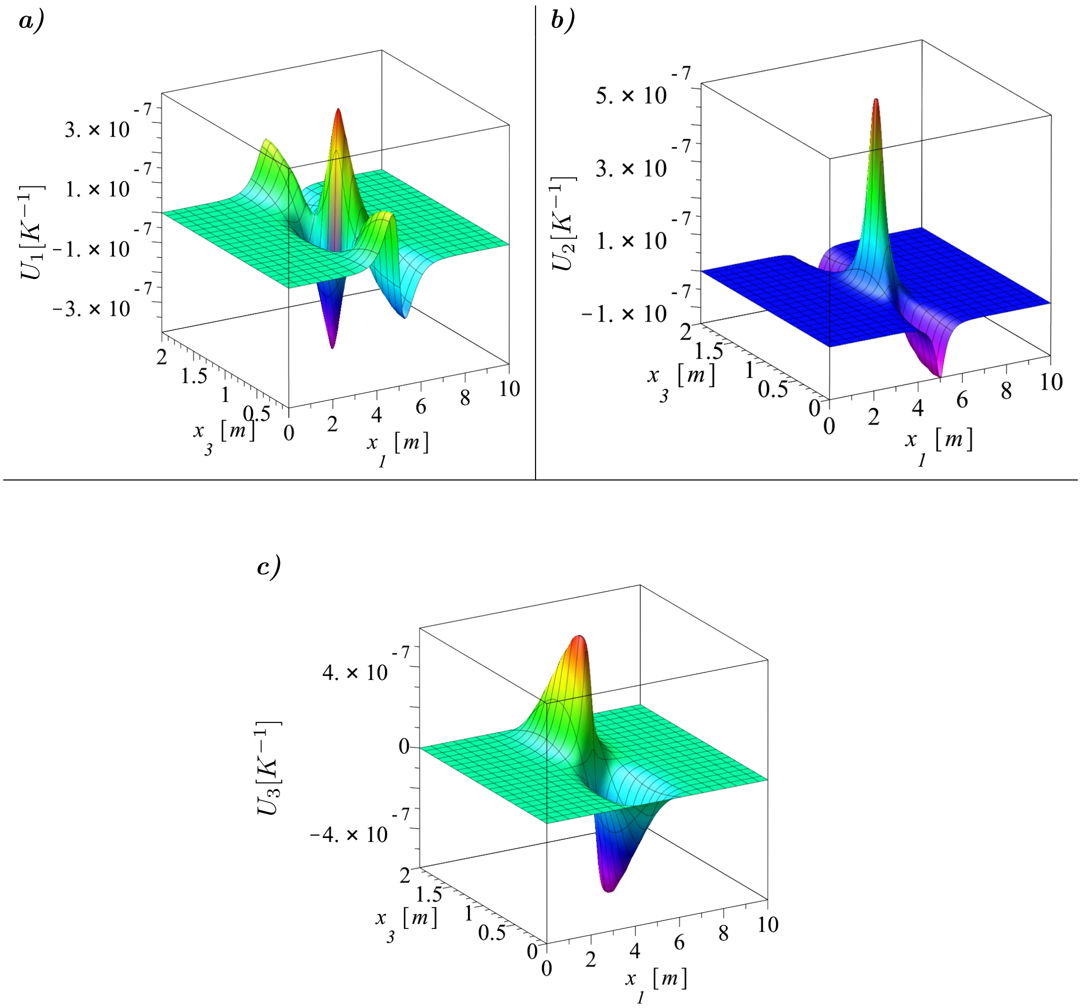

Figure 2. Graphs of MTDGFs $U_i(x, \xi)$ within the thermoelastic quarter of layer V for $0 \leq x_1 \leq 10m$; $x_2 = 2,1m$; $0 \leq x_3 \leq 2m$, caused by a unit heat source applied in the point $\xi_1 = 5m, \xi_2 = 2$ and $\xi_3 = 1m$.

In the Figure 2 can be observed:

1) All graphs were plotted by the soft Maple 18;
2) In the Figure 2, all graphs have jumps in the point $\xi_1 = 5m, \xi_2 = 2, \xi_3 = 1m$ of application of the unit point heat source. In this point the MTDGFs achieve maximal values;
3) All graphs of MTDGFs at infinity vanish;
4) The graph in the Figure 2*a)* is symmetrical in rapport with the plane $x_3 = 1m$. The boundary condition $U_1 = 0$ for $x_1 = 0$ is met (see eqn. (2));
5) The graph in the Figure 2*b)* is symmetrical in rapport with the planes $x_3 = 1m$;
6) The graph in the Figure 2*c)* is asymmetrical in rapport to the plane $x_3 = 1m$. The boundary conditions $U_3 = 0$ for $x_3 = 0$ and $x_3 = 2m$ are met (see eqns. (6);(8)).

CONCLUSION

On the base of analytical expressions for TVD and MTDGFs given in the Eqs. (45) - (48) (**see Example**) for BVP in Eqs (1), (2), (4), (6), (8) and respective IFs the readers will be able to obtain many particular solutions of 3D BVPs for a thermoelastic quarter of layer. Analogical results for TVD and MTDGFs can be obtained by them for remained fifteen combinations of the boundary conditions.

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